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Asymptotic behavior of solutions to the drift-diffusion equation of elliptic type

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1. Introduction

We consider the following initial-value problem for the drift-diffusion equation:

$$(1.1)\qquad \begin{cases} \partial_t u + (-\Delta)^{\theta/2} u - \nabla\cdot(u\nabla\psi) = 0, & t>0,\ x\in\mathbb{R}^n,\\ -\Delta\psi = u, & t>0,\ x\in\mathbb{R}^n,\\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases}$$

where $n \geq 2,\ 1 \leq \theta \leq 2,\ \partial_t = \partial/\partial t,\ (-\Delta)^{\theta/2}\varphi = \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}[\varphi]],\ \nabla = (\partial_1,\ldots,\partial_n),\ \partial_j = \partial/\partial x_j\ (1 \leq j \leq n),\ \Delta = \partial_1^2 + \cdots + \partial_n^2$, and $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a given initial data. If we put

$$u_\lambda(t,x) = \lambda^\theta u(\lambda^\theta t, \lambda x), \quad \psi_\lambda(t,x) = \lambda^{\theta-2}\psi(\lambda^\theta t, \lambda x)$$

for $\lambda > 0$ and solutions (u,ψ) to the drift-diffusion equation, then $(u_\lambda, \psi_\lambda)$ fulfill the equation, and

$$\sup_{t>0} \|u_\lambda(t)\|_{L^{n/\theta}(\mathbb{R}^n)} = \sup_{t>0} \|u(t)\|_{L^{n/\theta}(\mathbb{R}^n)}$$

for $\lambda > 0$. Particularly, when $1 < \theta < n$, it follows that

$$\sup_{t>0} \|\nabla\psi_\lambda(t)\|_{L^{n/(\theta-1)}(\mathbb{R}^n)} = \sup_{t>0} \|\nabla\psi(t)\|_{L^{n/(\theta-1)}(\mathbb{R}^n)}$$

for $\lambda > 0$, and Hardy-Littlewood-Sobolev's inequality leads that

$$(1.2)\qquad \|\nabla\psi(t)\|_{L^{n/(\theta-1)}(\mathbb{R}^n)} \leq C\|u(t)\|_{L^{n/\theta}(\mathbb{R}^n)}.$$

Therefore, we can treat solutions on the scale-invariant class

$$C\left((0,T), L^{n/\theta}(\mathbb{R}^n)\right)$$

whenever $1 < \theta < n$. But we call the case $\theta = 1$ the critical since (1.2) does not work. Though well-posedness in several classes, and global in time existence of solutions of (1.1) for $1 \leq \theta \leq 2$ were proved (see [10, 11, 12, 13, 15, 18]). Moreover, the solution satisfies

$$(1.3)\qquad u \in C^\infty\big((0,\infty), C^\infty(\mathbb{R}^n)\big),$$

and

$$(1.4)\qquad \|u(t)\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{\theta}(1-\frac{1}{p})}$$

for $1 \leq p \leq \infty$. The purpose here is to establish large-time behavior of the solution. When $1 < \theta \leq 2$, the L^p-theory for a parabolic equation yields the asymptotic expansion of the solution as the time variable tends to infinity (cf. [1, 9, 17]). The similar argument as in the above preceding works is effective on the several problems (see for example [2, 3, 4, 5, 7, 8, 16]). However, for (1.1) with $\theta = 1$, the L^p-theory for a parabolic equation does not work since the dissipation balances the nonlinearity. Thus the drift-diffusion equation with $\theta = 1$ is an equation of elliptic type. Throughout this paper, we study (1.1) with $\theta = 1$. Before stating our main theorems, we refer to the following generalized Burgers equation of elliptic type:

$$
\begin{cases} \partial_t \omega + (-\partial_x^2)^{1/2}\omega + \omega\partial_x\omega = 0, & t > 0, \ x \in \mathbb{R}, \\ \omega(0,x) = \omega_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.5}
$$

For global solutions of (1.5), Iwabuchi [6] established the asymptotic expansion by employing the corresponding Besov spaces (see Section 4). To discuss large-time behavior of the solution of (1.1), we introduce the following integral equation associated with (1.1):

$$
u(t) = P(t) * u_0 + \int_0^t P(t-s) * \nabla \cdot (u\nabla(-\Delta)^{-1}u)(s)ds, \tag{1.6}
$$

where the Poisson kernel

$$
P(t,x) = \pi^{-\frac{n+1}{2}}\Gamma(\tfrac{n+1}{2})\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}
$$

is the fundamental solution to $\partial_t u + (-\Delta)^{\theta/2}u = 0$, and $*$ denotes the convolution for x. The solution of (1.6) is called the mild solution and solves (1.1).

Theorem 1.1 ([20]). *Let $n \geq 3$, $\theta = 1$, $u_0 \in L^1(\mathbb{R}^n, \sqrt{1+|x|^2}dx)$, and the solution u of* (1.1) *satisfy* (1.3) *and* (1.4). *Then*

$$
\|u(t) - M_u P(t) - m_u \cdot \nabla P(t)\|_{L^p(\mathbb{R}^n)} = o\big(t^{-n(1-\frac{1}{p})-1}\big)
$$

as $t \to \infty$ for any $1 < p < \infty$, where $M_u = \int_{\mathbb{R}^n} u_0(y)dy$ and $m_u = \int_{\mathbb{R}^n}(-y)u_0(y)dy$.

In the two-dimensional case, we introduce the following function:

$$
J(t,x) = \int_0^t P(t-s) * \nabla \cdot (P\nabla(-\Delta)^{-1}P)(s)ds. \tag{1.7}
$$

This function is well-defined, and satisfies

$$
J \in C\big((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\big),
$$

and

$$
\|J(t)\|_{L^p(\mathbb{R}^2)} = t^{-2(1-\frac{1}{p})-1}\|J(1)\|_{L^p(\mathbb{R}^2)}
$$

for $1 \leq p \leq \infty$ and $t > 0$. We remark that this decay rate is same as one of $\nabla P(t)$. Then we obtain the asymptotic expansion for (1.1) with $n = 2$.

Theorem 1.2 ([20]). *Let $n = 2$, $\theta = 1$, $u_0 \in L^1(\mathbb{R}^2, \sqrt{1+|x|^2}dx)$, and the solution u of* (1.1) *satisfy* (1.3) *and* (1.4). *Then*

$$
\big\|u(t) - M_u P(t) - m_u \cdot \nabla P(t) - M_u^2 J(t)\big\|_{L^p(\mathbb{R}^2)} = o\big(t^{-2(1-\frac{1}{p})-1}\big)
$$

as $t \to \infty$ for any $1 < p < \infty$, where $M_u = \int_{\mathbb{R}^2} u_0(y)dy$ and $m_u = \int_{\mathbb{R}^2}(-y)u_0(y)dy$.

Since $J(t)$ corrects the asymptotic expansion, we call this function the correction term. The proofs of Theorems 1.1 and 1.2 are based on the L^p-L^q type estimate for (1.6) with the aid of the energy method.

2. Preliminaries

Hardy-Littlewood-Sobolev's inequality yields the following inequality.

Lemma 2.1. *Let $n \geq 2$, $1 < \sigma < n$, $1 < p < \frac{n}{\sigma}$ and $\frac{1}{p_*} = \frac{1}{p} - \frac{\sigma}{n}$. Then there exists a positive constant C such that*

$$\|(-\Delta)^{-\sigma/2}\varphi\|_{L^{p_*}(\mathbb{R}^n)} \leq C\|\varphi\|_{L^p(\mathbb{R}^n)}$$

for any $\varphi \in L^p(\mathbb{R}^n)$.

It follows that

$$\|\nabla(-\Delta)^{-1}\varphi\|_{L^\infty(\mathbb{R}^n)} \leq C\left((1+t)\|\varphi\|_{L^\infty(\mathbb{R}^n)} + (1+t)^{-n+1}\|\varphi\|_{L^1(\mathbb{R}^n)}\right) \tag{2.1}$$

for any $\varphi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $t > 0$. Indeed

$$|\nabla(-\Delta)^{-1}\varphi(x)| \leq C\left(\int_{|x-y|\leq 1+t} + \int_{|x-y|>1+t}\right)\frac{|\varphi(y)|}{|x-y|^{n-1}}dy$$

immediately gives (2.1). Since the Poisson kernel fulfills

$$\|\partial_t^k(-\Delta)^{\sigma/2}P(t)\|_{L^p(\mathbb{R}^n)} = t^{-n(1-\frac{1}{p})-k-\sigma}\|\partial_t^k\nabla^\alpha P(1)\|_{L^p(\mathbb{R}^n)}$$

for $k \in \mathbb{Z}_+$, $\sigma \geq 0$, $1 \leq p \leq \infty$ and $t > 0$, we obtain the following lemmas.

Lemma 2.2. *Let $n \geq 1$, $1 \leq p \leq q \leq \infty$, $k \in \mathbb{Z}_+$ and $\sigma \geq 0$. Then there exists a positive constant C such that*

$$\|\partial_t^k(-\Delta)^{\sigma/2}P(t) * \varphi\|_{L^q(\mathbb{R}^n)} \leq Ct^{-n(\frac{1}{p}-\frac{1}{q})-k-\sigma}\|\varphi\|_{L^p(\mathbb{R}^n)}$$

for any $\varphi \in L^p(\mathbb{R}^n)$ and $t > 0$.

Lemma 2.3. *Let $n \geq 1$, $k \in \mathbb{Z}_+$ and $\varphi \in L^1(\mathbb{R}^n, (1+|x|^2)^{k/2}\,dx)$. Then*

$$\left\|P(t) * \varphi - \sum_{|\alpha|\leq k}\frac{\nabla^\alpha P(t)}{\alpha!}\int_{\mathbb{R}^n}(-y)^\alpha\varphi(y)dy\right\|_{L^p(\mathbb{R}^n)} = o\big(t^{-n(1-\frac{1}{p})-k}\big)$$

as $t \to \infty$ for any $1 \leq p \leq \infty$. In addition, if $|x|^{k+1}\varphi \in L^1(\mathbb{R}^n)$, then

$$\left\|P(t) * \varphi - \sum_{|\alpha|\leq k}\frac{\nabla^\alpha P(t)}{\alpha!}\int_{\mathbb{R}^n}(-y)^\alpha\varphi(y)dy\right\|_{L^p(\mathbb{R}^n)} \leq Ct^{-n(1-\frac{1}{p})-k}(1+t)^{-1}$$

for any $1 \leq p \leq \infty$ and $t > 0$.

Proposition 2.4. *Let $n \geq 2$, $\theta = 1$, and the solution u of (1.1) satisfy (1.3) and (1.4). Then there exist positive constants C and T such that*

$$\|(-\Delta)^{1/4}u(t)\|_{L^2(\mathbb{R}^n)} \leq Ct^{-1/2}(1+t)^{-n/2} \tag{2.2}$$

for any $t \geq T$.

Proof. We multiply (1.1) by $t^q(-\Delta)^{1/2}u$ for sufficiently large $q > 0$, and have

$$\begin{aligned}
&\frac{d}{dt}\left(t^q\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R}^n)}^2\right) + 2t^q\|(-\Delta)^{1/2}u\|_{L^2(\mathbb{R}^n)}^2 \\
&= -t^q\int_{\mathbb{R}^n}\nabla u\cdot\nabla(-\Delta)^{-1}u(-\Delta)^{1/2}udx + t^q\int_{\mathbb{R}^n}u^2(-\Delta)^{1/2}udx \\
&\quad + qt^{q-1}\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned} \tag{2.3}$$

By using (2.1) and (1.4), we see that

$$
\begin{aligned}
t^q\left|\int_{\mathbb{R}^3} \nabla u \cdot \nabla(-\Delta)^{-1}u(-\Delta)^{1/2}u dx\right| \leq & Ct^q\left\|\nabla(-\Delta)^{-1}u(t)\right\|_{L^\infty(\mathbb{R}^n)}\left\|(-\Delta)^{1/2}u\right\|^2_{L^2(\mathbb{R}^n)} \\
\leq & \frac{1}{3}t^q\left\|(-\Delta)^{1/2}u\right\|^2_{L^2(\mathbb{R}^n)}
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
t^q\left|\int_{\mathbb{R}^n} u^2(-\Delta)^{1/2}u dx\right| \leq & t^q\left\|u\right\|^2_{L^4(\mathbb{R}^n)}\left\|(-\Delta)^{1/2}u\right\|_{L^2(\mathbb{R}^n)} \\
\leq & Ct^q(1+t)^{-3n} + \frac{1}{3}t^q\left\|(-\Delta)^{1/2}u\right\|^2_{L^2(\mathbb{R}^n)}
\end{aligned} \tag{2.5}
$$

for large $t > 0$. Gagliardo-Nirenberg's inequality and (1.4) lead that

$$
\begin{aligned}
qt^{q-1}\left\|(-\Delta)^{1/4}u\right\|^2_{L^2(\mathbb{R}^n)} \leq & Ct^{q-1}\left\|u\right\|_{L^2(\mathbb{R}^n)}\left\|(-\Delta)^{1/2}u\right\|_{L^2(\mathbb{R}^n)} \\
\leq & Ct^{q-1}(1+t)^{-n/2}\left\|(-\Delta)^{1/2}u\right\|_{L^2(\mathbb{R}^n)} \\
\leq & Ct^{q-2}(1+t)^{-n} + \frac{1}{3}t^q\left\|(-\Delta)^{1/2}u\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{2.6}
$$

By applying (2.4), (2.5) and (2.6) to (2.3), we complete the proof. □

3. Outline of the Proof of Main Results

In this section, we outline the proof of our main theorem. The detailed proofs will appear in [20]. Before proving Theorem 1.2, we prepare the following two propositions.

Proposition 3.1. *Let $n = 2$, $\theta = 1$, $u_0 \in L^1(\mathbb{R}^2, \sqrt{1+|x|^2}\,dx)$, and the solution u of* (1.1) *satisfy* (1.3) *and* (1.4). *Assume that $1 < p < \infty$. Then*

$$
\|u(t) - M_u P(t)\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-\frac{1}{p})}(1+t)^{-1}\log(2+t) \tag{3.1}
$$

for any $t > 0$.

Proposition 3.2. *Let $n = 2$, $\theta = 1$, $u_0 \in L^1(\mathbb{R}^2, \sqrt{1+|x|^2}\,dx)$, and the solution u of* (1.1) *satisfy* (1.3) *and* (1.4). *Assume that $1 < p < \infty$ and $0 < \sigma < \frac{1}{4p}$. Then there exist positive constants C and T such that*

$$
\left\|(-\Delta)^{\sigma/2}\left(u(t) - M_u P(t)\right)\right\|_{L^p(\mathbb{R}^2)} \leq Ct^{-2(1-\frac{1}{p})-\sigma}(1+t)^{-1}\log(2+t)
$$

for any $t \geq T$.

The above propositions are proved by the L^p-L^q estimate for (1.6) together with (1.4) and (2.2).

Outline of Theorem 1.2. From (1.6) and (1.7), we see

$$
\begin{aligned}
&u(t) - M_uP(t) - m_u \cdot \nabla P(t) - M_u^2 J(t)\\
=&P(t) * u_0 - M_uP(t) - m_u \cdot \nabla P(t)\\
&+ \int_0^{t/2} \nabla P(t-s) * \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s)\big)\, ds\\
&+ M_u^2 \int_0^{t/2} \nabla P(t-s) * \big((P\nabla(-\Delta)^{-1}P)(1+s) - (P\nabla(-\Delta)^{-1}P)(s)\big)\, ds\\
&+ \int_{t/2}^{t} P(t-s) * \nabla \cdot \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2(P\nabla(-\Delta)^{-1}P)(s)\big)\, ds.
\end{aligned} \tag{3.2}
$$

Lemma 2.3 gives that

$$
\left\|P(t) * u_0 - M_uP(t) - m_u \cdot \nabla P(t)\right\|_{L^p(\mathbb{R}^2)} = o\left(t^{-2(1-\frac{1}{p})-1}\right) \tag{3.3}
$$

as $t \to \infty$. The second term on the right-hand side is rewritten by

$$
\begin{aligned}
&\int_0^{t/2} \nabla P(t-s) * \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s)\big)\, ds\\
=&\int_0^{t/2}\int_{\mathbb{R}^2} (\nabla P(t-s,x-y) - \nabla P(t-s,x))\\
&\quad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, dyds
\end{aligned}
$$

since $\int_{\mathbb{R}^2}((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y))dy = 0$ for $s > 0$. We introduce $R(t) = o(t)$ as $t \to \infty$, then, by Taylor's theorem, we see that

$$
\begin{aligned}
&\int_0^{t/2}\int_{\mathbb{R}^2} (\nabla P(t-s,x-y) - \nabla P(t-s,x))\\
&\quad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, dyds\\
=&\int_0^{t/2}\int_{|y|\le R(t)}\int_0^1 (-y\cdot\nabla)\nabla P(t-s,x-y+\lambda y)\\
&\quad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, d\lambda dyds\\
&+\int_0^{t/2}\int_{|y|> R(t)} (\nabla P(t-s,x-y) - \nabla P(t-s,x))\\
&\quad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, dyds.
\end{aligned}
$$

By employing Lemma 2.2 together with (1.4), Lemma 2.1 and Proposition 3.1, we have that

$$
\begin{aligned}
&\Bigg\|\int_0^{t/2}\int_{|y|\le R(t)}\int_0^1 (-y\cdot\nabla)\nabla P(t-s,x-y+\lambda y)\\
&\quad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2(P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, d\lambda dyds\Bigg\|_{L^p(\mathbb{R}^2)}\\
\le& CR(t)\int_0^{t/2} (t-s)^{-(1-\frac{1}{p})-2}(1+s)^{-2} ds = o\big(t^{-2(1-\frac{1}{p})-1}\big)
\end{aligned}
$$

as $t \to \infty$. Similarly, we obtain that

$$\begin{aligned}&\Big\| \int_0^{t/2} \int_{\mathbb{R}^2} (\nabla P(t-s, x-y) - \nabla P(t-s, x)) \\ &\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, dyds \Big\|_{L^p(\mathbb{R}^2)} \\ &= O\big(t^{-2(1-\frac{1}{p})-1}\big)\end{aligned}$$

as $t \to \infty$. Hence Lebesgue's monotone theorem yields that

$$\begin{aligned}&\Big\| \int_0^{t/2} \int_{|y|>R(t)} (\nabla P(t-s, x-y) - \nabla P(t-s, x)) \\ &\qquad \cdot \big((u\nabla(-\Delta)^{-1}u)(s,y) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s,y)\big)\, dyds \Big\|_{L^p(\mathbb{R}^2)} \\ &= o\big(t^{-2(1-\frac{1}{p})-1}\big)\end{aligned}$$

as $t \to \infty$. Therefore, it follows that

$$\begin{aligned}&\Big\| \int_0^{t/2} \nabla P(t-s) * \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(1+s)\big)\, ds \Big\|_{L^p(\mathbb{R}^2)} \\ &= o\big(t^{-2(1-\frac{1}{p})-1}\big)\end{aligned} \tag{3.4}$$

as $t \to \infty$. We see at once that

$$\begin{aligned}&\Big\| \int_0^{t/2} \nabla P(t-s) * \big((P\nabla(-\Delta)^{-1}P)(1+s) - (P\nabla(-\Delta)^{-1}P)(s)\big)\, ds \Big\|_{L^p(\mathbb{R}^2)} \\ &= o\big(t^{-2(1-\frac{1}{p})-1}\big)\end{aligned} \tag{3.5}$$

as $t \to \infty$. We represent the last term on the right-hand side of (3.2) by

$$\begin{aligned}&\int_{t/2}^{t} P(t-s) * \nabla \cdot \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(s)\big)\, ds \\ &= \int_{t/2}^{t} \nabla(-\Delta)^{-\sigma/2} P(t-s) * (-\Delta)^{\sigma/2} \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(s)\big)\, ds\end{aligned}$$

for some small $\sigma > 0$. Thus, by employing Lemma 2.2 together with Lemma 2.1, (2.2) and Proposition 3.2, we conclude that

$$\begin{aligned}&\Big\| \int_{t/2}^{t} P(t-s) * \nabla \cdot \big((u\nabla(-\Delta)^{-1}u)(s) - M_u^2 (P\nabla(-\Delta)^{-1}P)(s)\big)\, ds \Big\|_{L^p(\mathbb{R}^2)} \\ &\leq C \int_{t/2}^{t} (t-s)^{-(1-\sigma)} s^{-2(1-\frac{1}{p})-2-\sigma} ds = o\big(t^{-2(1-\frac{1}{p})-1}\big)\end{aligned} \tag{3.6}$$

as $t \to \infty$. By applying (3.3), (3.4), (3.5) and (3.6) to (3.2), we complete the outline. □

Theorem 1.1 is proved in the similar way.

4. The Drift-Diffusion Equation and the Burgers Equation

We expect that the solution of the two dimensional drift-diffusion equation and one of the Burgers equation have a similar decay structure since those nonlinear terms decay with same order. Namely

$$\|\omega\partial_x\omega(t)\|_{L^1(\mathbb{R})} = O(t^{-1})$$

and

$$\|u\nabla(-\Delta)^{-1}u(t)\|_{L^1(\mathbb{R}^2)} = O(t^{-1})$$

as $t \to \infty$. To discuss an asymptotic expansion for (1.5) we make the following definition:

$$\begin{aligned} J_\omega(t,x) = & -\frac{1}{2}\int_0^{t/2}\int_{\mathbb{R}} \left(\partial_x P(t-s,x-y) - \partial_x P(t,x)\right) P(s,y)^2 dyds \\ & - \int_{t/2}^t P(t-s) * (P\partial_x P)(s)ds. \end{aligned} \tag{4.1}$$

This function is well-defined in $C((0,\infty), L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, and satisfies

$$\|J_\omega(t)\|_{L^p(\mathbb{R})} = t^{-(1-\frac{1}{p})-1}\|J_\omega(1)\|_{L^p(\mathbb{R})}$$

for any $1 \le p \le \infty$ and $t > 0$. Then, for $1 \le p \le \infty$, the decaying solution $\omega(t)$ of (1.5) fulfills

$$\begin{aligned} & \Bigg\| \omega(t) - M_\omega P(t) + \frac{1}{4\pi}M_\omega^2 \partial_x P(t)\log(2+t) - M_\omega^2 J_\omega(t) \\ & - \left(m_\omega - \frac{1}{2}\int_0^\infty \int_{\mathbb{R}} \left(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2\right) dyds + \frac{1}{4\pi}M_\omega^2 \log 2\right)\partial_x P(t)\Bigg\|_{L^p(\mathbb{R})} \\ = & o\left(t^{-(1-\frac{1}{p})-1}\right) \end{aligned}$$

as $t \to \infty$, where $M_\omega = \int_{\mathbb{R}} \omega_0(y)dy$ and $m_\omega = \int_{\mathbb{R}}(-y)\omega_0(y)dy$ (cf. [6, 20]). The logarithmic term on this is derived from the following procedure: The mild solution of (1.5) is given by

$$\omega(t) = P(t) * \omega_0 - \frac{1}{2}\int_0^{t/2} \partial_x P(t-s) * (\omega^2)(s)ds - \int_{t/2}^t P(t-s) * (\omega\partial_x\omega)(s)ds. \tag{4.2}$$

We rewrite the nonlinear term by

$$\begin{aligned} & \int_0^{t/2} \partial_x P(t-s) * (\omega^2)(s)ds \\ = & \int_0^{t/2} \partial_x P(t-s) * \left(\omega(s)^2 - M_\omega^2 P(1+s)^2\right) ds + M_\omega^2 \int_0^{t/2} \partial_x P(t-s) * (P^2)(1+s)ds \\ = & \partial_x P(t) \int_0^\infty \int_{\mathbb{R}} \left(\omega(s)^2 - M_\omega^2 P(1+s,y)^2\right) dyds + M_\omega^2 \partial_x P(t) \int_0^{t/2}\int_{\mathbb{R}} P(1+s,y)^2 dyds \\ & + M_\omega^2 \int_0^{t/2}\int_{\mathbb{R}} \left(\partial_x P(t-s,x-y) - \partial_x P(t,x)\right) P(s,y)^2 dyds + \rho_1(t) + \rho_2(t) + \rho_3(t), \end{aligned}$$

where

$$\rho_1(t) = \int_0^{t/2}\int_{\mathbb{R}} (\partial_x P(t-s,x-y) - \partial_x P(t,x)) \left(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2\right) dyds,$$

$$\rho_2(t) = -\, \partial_x P(t,x) \int_{t/2}^{\infty}\int_{\mathbb{R}} \left(\omega(s,y)^2 - M_\omega^2 P(1+s,y)^2\right) dyds,$$

$$\rho_3(t) = M_\omega^2 \int_0^{t/2}\int_{\mathbb{R}} (\partial_x P(t-s,x-y) - \partial_x P(t,x)) \left(P(1+s,y)^2 - P(s,y)^2\right) dyds.$$

The third term on the right-hand side is a part of $J_\omega(t)$, and the second term will lead the logarithmic term. Indeed we see that $\int_0^{t/2}\int_{\mathbb{R}} P(1+s,y)^2 dyds = \int_0^{t/2}(1+s)^{-1}ds \int_{\mathbb{R}} P(1,y)^2 dy = \frac{1}{2\pi}(\log(2+t) - \log 2)$. Since $\omega(t)$ converges to $M_\omega P(t)$, we can confirm that

$$\|\rho_1(t)\|_{L^p(\mathbb{R})}, \|\rho_2(t)\|_{L^p(\mathbb{R})}, \|\rho_3(t)\|_{L^p(\mathbb{R})} = o(t^{-(1-\frac{1}{p})-1})$$

as $t \to \infty$. In the study for (1.1), the similar logarithmic term as above appears seemingly. Namely, in the same manner as above, the nonlinear term on (1.6) provides

$$\begin{aligned} &M_u^2 \nabla P(t) \cdot \int_0^{t/2}\int_{\mathbb{R}^2} (P\nabla(-\Delta)^{-1}P)(1+s,y)dyds \\ =&M_u^2 \nabla P(t) \cdot \int_0^{t/2} (1+s)^{-1} ds \int_{\mathbb{R}^2} (P\nabla(-\Delta)^{-1}P)(1,y)dy \end{aligned}$$

into the asymptotic expansion. However, since $\int_{\mathbb{R}^2}(P\nabla(-\Delta)^{-1}P)(1,y)dy = 0$, this term is vanishing.

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